

A PROPERLY INFINITE C*-ALGEBRA WHICH IS NOT K_1 -INJECTIVE

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ABSTRACT. We construct in this note a unital properly infinite C*-algebra which is not K_1 -injective.

In memory of Uffe Haagerup

1. NOTATIONS

The C*-algebra \mathcal{O}_2 defined by J. Cuntz in [Cun77] is the unital C*-algebra generated by two isometries s_1, s_2 satisfying the equality $s_1 s_1^* + s_2 s_2^* = 1$.

We call \mathcal{T}_2 the extension of \mathcal{O}_2 by the C*-algebra of compact operators acting on an infinite dimensional Hilbert space. This universal unital C*-algebra is generated by two isometries v_1, v_2 satisfying the inequality $v_1 v_1^* + v_2 v_2^* \leq 1$ ([Cun77]).

A nonzero unital C*-algebra A is said to be *properly infinite* if and only if there exists an homomorphism of unital C*-algebra $\mathcal{T}_2 \rightarrow A$ (see [Rør04, Proposition 2.1]).

A nonzero unital C*-algebra A is said to be *K_1 -injective* if and only if any unitary $u \in A$ satisfies $[u] = [1_A]$ in $K_1(A)$ only if it belongs to the connected component $\mathcal{U}^0(A)$ of the unit 1_A in the group $\mathcal{U}(A)$ of unitaries in A (see *e.g.* [BRR08, definition 2.6]).

We write $u_1 \sim_h u_2$ when two unitaries u_1, u_2 in $\mathcal{U}(A)$ satisfy $u_1 \cdot u_2^* \in \mathcal{U}^0(A)$.

We answer in Corollary 4 several questions from [BRR08], [Blan09], [Blan16] on the stability of proper infiniteness under continuous deformations.

2. MAIN RESULT

Let \mathbf{u} be the canonical unitary generator of the C*-algebra $C^*(\mathbb{Z})$ and denote by ι_0 and ι_1 the two canonical *-embeddings of the C*-algebra \mathcal{O}_2 in the full unital free product $\mathcal{O}_2 *_\mathbb{C} \mathcal{O}_2$. Then one has the following description for the K-theory of the unital free product $\mathcal{O}_2 *_\mathbb{C} \mathcal{O}_2$.

- Lemma 1.** 1) $K_0(\mathcal{O}_2 *_\mathbb{C} \mathcal{O}_2) = 0$ and $K_1(\mathcal{O}_2 *_\mathbb{C} \mathcal{O}_2) = \mathbb{Z}$.
 2) There is an isomorphism of unital C*-algebra $\mathcal{O}_2 *_\mathbb{C} C^*(\mathbb{Z}) \cong \mathcal{O}_2 *_\mathbb{C} \mathcal{O}_2$.
 3) The unitary \mathbf{u} has a generating image in $K_1(\mathcal{O}_2 *_\mathbb{C} C^*(\mathbb{Z}))$.

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Proof. 1) As noticed by B. Blackadar in [Blac07], Corollary 2.6 of [Ger97] implies the exactness of the six-term cyclic sequence of topological K-theory group

$$\begin{array}{ccccc} K_0(\mathbb{C}) = \mathbb{Z} & \longrightarrow & K_0(\mathcal{O}_2 \oplus \mathcal{O}_2) = 0 \oplus 0 & \longrightarrow & K_0(\mathcal{O}_2 *_\mathbb{C} \mathcal{O}_2) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{O}_2 *_\mathbb{C} \mathcal{O}_2) & \longleftarrow & K_1(\mathcal{O}_2 \oplus \mathcal{O}_2) = 0 \oplus 0 & \longleftarrow & K_1(\mathbb{C}) = 0 \end{array}$$

2) The unital free product $\mathcal{O}_2 *_\mathbb{C} C^*(\mathbb{Z})$ is generated by the two isometries s_1, s_2 and the unitary \mathbf{u} whereas the unital free product $\mathcal{O}_2 *_\mathbb{C} \mathcal{O}_2$ is generated by the four isometries $\iota_0(s_1), \iota_0(s_2), \iota_1(s_1), \iota_1(s_2)$. The unique C^* -morphism $\sigma : \mathcal{O}_2 *_\mathbb{C} C^*(\mathbb{Z}) \rightarrow \mathcal{O}_2 *_\mathbb{C} \mathcal{O}_2$ with

$$\sigma(s_k) = \iota_0(s_k) \quad (k = 1, 2) \quad \text{and} \quad \sigma(\mathbf{u}) = \iota_1(s_1)\iota_0(s_1^*) + \iota_1(s_2)\iota_0(s_2^*)$$

is an isomorphism satisfying $\sigma^{-1}(\iota_0(s_k)) = s_k$ and $\sigma^{-1}(\iota_1(s_k)) = \mathbf{u} \cdot s_k$ for $k = 1, 2$.

3) The unitary \mathbf{u} generates a canonical copy of the C^* -algebra $C^*(\mathbb{Z})$ inside the unital free product $\mathcal{O}_2 *_\mathbb{C} C^*(\mathbb{Z})$ and corollary 2.6 of [Ger97] entails the exactness of the cyclic sequence

$$\begin{array}{ccccc} K_0(\mathbb{C}) = \mathbb{Z} & \longrightarrow & K_0(\mathcal{O}_2 \oplus C^*(\mathbb{Z})) = 0 \oplus \mathbb{Z} & \longrightarrow & K_0(\mathcal{O}_2 *_\mathbb{C} C^*(\mathbb{Z})) = 0 \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{O}_2 *_\mathbb{C} C^*(\mathbb{Z})) = \mathbb{Z} & \longleftarrow & K_1(\mathcal{O}_2 \oplus C^*(\mathbb{Z})) = 0 \oplus \mathbb{Z} & \longleftarrow & K_1(\mathbb{C}) = 0 \end{array}$$

The injectivity of the upper left arrow implies that the lower left arrow is an isomorphism and the class $[\mathbf{u}]$ is a generator of the group $K_1(\mathcal{O}_2 *_\mathbb{C} C^*(\mathbb{Z})) \cong \mathbb{Z}$. \square

Denote by j_0 and j_1 the two canonical embeddings of the C^* -algebra \mathcal{T}_2 in the unital free product $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$. As the projection $1 - v_1 v_1^*$ is properly infinite and full in \mathcal{T}_2 , the two projections $j_0(1 - v_1 v_1^*), j_1(1 - v_1 v_1^*)$ are properly infinite and full in $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$ and there exists (Lemma 2.4 in [BRR08]) a unitary $\tilde{u} \in \mathcal{U}(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2)$ with $j_1(v_1) = \tilde{u} \cdot j_0(v_1)$.

Define also the K_1 -trivial unitary \dot{u} in the quotient $\mathcal{O}_2 *_\mathbb{C} \mathcal{O}_2 = \mathcal{O}_2 *_\mathbb{C} C^*(\mathbb{Z})$ by

$$\begin{aligned} \dot{u} &= [\iota_1(s_1)\iota_0(s_1^*) + \iota_1(s_2)\iota_0(s_2^*)] \cdot [\iota_0(s_1 s_1^*) + \iota_0(s_2)[\iota_0(s_1)\iota_1(s_1^*) + \iota_0(s_2)\iota_1(s_2^*)]\iota_0(s_2^*)] \\ &= \mathbf{u} \cdot (s_1 s_1^* + s_2 \mathbf{u} s_2^*)^{-1}. \end{aligned}$$

Then the unital free product $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$ enjoys the following properties.

Proposition 2. 1) $K_0(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2) = \mathbb{Z}$ and $K_1(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2) = 0$.

2) The direct sum $\tilde{u} \oplus 1_{\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2}$ belongs to the connected component $\mathcal{U}^0(M_2(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2))$.

3) The unitary \tilde{u} is in $\mathcal{U}^0(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2)$ if and only if the unitary \dot{u} is in $\mathcal{U}^0(\mathcal{O}_2 *_\mathbb{C} C^*(\mathbb{Z}))$.

4) Any unital $*$ -representation Θ of the unital free product $\mathcal{O}_2 *_\mathbb{C} C^*(\mathbb{Z})$ which factors through the tensor product $\mathcal{O}_2 \otimes C^*(\mathbb{Z})$ satisfies $\Theta(\dot{u}) \in \mathcal{U}^0(\Theta(\mathcal{O}_2 *_\mathbb{C} C^*(\mathbb{Z})))$.

5) The unitary \dot{u} is not in the connected component $\mathcal{U}^0(\mathcal{O}_2 *_\mathbb{C} C^*(\mathbb{Z}))$.

6) The properly infinite C^* -algebra $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$ is not K_1 -injective.

Proof. 1) The suspension $C_0((0, 1)) \otimes \mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$ is KK -equivalent to the cone $D = \{(f_0, f_1) \in C_0((0, 1], \mathcal{T}_2 \oplus \mathcal{T}_2), f_0(1) = f_1(1) \in \mathbb{C} \cdot 1_{\mathcal{T}_2}\}$ (Theorem 2.2 in [Ger97]) and that cone is KK -equivalent to the C^* -algebra $\{(f_0, f_1) \in C((0, 1])^2, f_0(1) = f_1(1)\} \cong C_0((0, 2))$. Thus, $K_i(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2) = K_{1-i}(D) = K_{1-i}(C_0(0, 2))$ for $i = 0, 1$.

2) The two projections $0 \oplus 1$ and $1 \oplus 0$ are properly infinite and full in $M_2(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2)$. Hence, the unitary $\tilde{u} \oplus 1$ belongs to $\mathcal{U}^0(M_2(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2))$ (see [LLR00, exercise 8.11]).

3) Call π the unique quotient morphism of unital C^* -algebra from the full unital free product $\mathcal{T}_2 *_C \mathcal{T}_2$ to $\mathcal{O}_2 *_C \mathcal{O}_2$ such that $\pi(j_i(v_k)) = \iota_i(s_k)$ for all $i = 0, 1$ and $k = 1, 2$.

Then $\pi(\tilde{u})s_1 = \pi(\tilde{u}j_0(v_1)) = \pi(j_1(v_1)) = \mathbf{u}s_1$ and the product $\pi(\tilde{u}) \cdot (\mathring{u})^*$ satisfies

$$\begin{aligned} \pi(\tilde{u}) \cdot (\mathring{u})^* \cdot \mathbf{u}s_1s_1^*\mathbf{u}^* &= \pi(\tilde{u}) \cdot s_1 \cdot s_1^*\mathbf{u}^* \\ &= \mathbf{u}s_1 \cdot s_1^*\mathbf{u}^* \quad \text{in } \mathcal{O}_2 *_C C^*(\mathbb{Z}). \end{aligned}$$

Accordingly, the product $\pi(\tilde{u}) \cdot (\mathring{u})^*$ is a K_1 -trivial unitary in $\mathcal{O}_2 *_C C^*(\mathbb{Z})$ which commutes to the projection $\mathbf{u}s_1s_1^*\mathbf{u}^*$. As both $\mathbf{u}s_1s_1^*\mathbf{u}^*$ and $1 - \mathbf{u}s_1s_1^*\mathbf{u}^*$ are properly infinite full projections in $\mathcal{O}_2 *_C C^*(\mathbb{Z})$, Lemma 2.4 of [BRR08] implies that $\pi(\tilde{u}) \cdot (\mathring{u})^*$ belongs to $\mathcal{U}^0(\mathcal{O}_2 *_C C^*(\mathbb{Z}))$ and so $\pi(\tilde{u}) = \pi(\tilde{u}) \cdot (\mathring{u})^* \cdot \mathring{u} \sim_h \mathring{u}$ in $\mathcal{U}(\mathcal{O}_2 *_C C^*(\mathbb{Z}))$.

– If $\tilde{u} \sim_h 1_{\mathcal{T}_2 *_C \mathcal{T}_2}$ in $\mathcal{U}(\mathcal{T}_2 *_C \mathcal{T}_2)$, then $\mathring{u} \sim_h \pi(\tilde{u}) \sim_h \pi(1_{\mathcal{T}_2 *_C \mathcal{T}_2}) = 1_{\mathcal{O}_2 *_C \mathcal{O}_2}$ in $\mathcal{U}(\mathcal{O}_2 *_C \mathcal{O}_2)$.

– If $\mathring{u} \sim_h 1_{\mathcal{O}_2 *_C \mathcal{O}_2}$ in $\mathcal{U}(\mathcal{O}_2 *_C \mathcal{O}_2)$, then the homotopic unitary $\pi(\tilde{u})$ also belongs to the connected component $\mathcal{U}^0(\mathcal{O}_2 *_C \mathcal{O}_2)$ and its lift \tilde{u} in $\mathcal{U}(\mathcal{T}_2 *_C \mathcal{T}_2)$ can be continuously connected to the unit $1_{\mathcal{T}_2 *_C \mathcal{T}_2}$ (see *e.g.* Lemma 2.1.7 in [LLR00]).

4) Let $\sigma : \mathcal{O}_2 *_C C^*(\mathbb{Z}) = C^*\langle s_1, s_2, \mathbf{u} \rangle \rightarrow \mathcal{O}_2 \otimes C^*(\mathbb{Z}) = C(S^1, \mathcal{O}_2)$ be the unique C^* -epimorphism satisfying the relations

$$\sigma(s_k) = s_k \otimes 1 \quad (k = 1, 2) \quad \text{and} \quad \sigma(\mathbf{u}) = 1 \otimes \mathbf{u}.$$

The two isometries $V_1 = (s_1s_1^* + s_2s_1s_2^*) \otimes 1$ and $V_2 = s_2s_2 \otimes 1$ generate a unital copy of \mathcal{O}_2 in $\mathcal{O}_2 \otimes C^*(\mathbb{Z})$ and we have in $\mathcal{U}(\mathcal{O}_2 \otimes C^*(\mathbb{Z}))$ the sequence of homotopies

$$\begin{aligned} \sigma(\mathring{u}) &= \sigma(\mathbf{u}(s_1s_1^* + s_2\mathbf{u}^*s_2^*)) \\ &= s_1s_1^* \otimes \mathbf{u} + (1 \otimes \mathbf{u})(s_2 \otimes 1)(1 \otimes \mathbf{u}^*)(s_2^* \otimes 1) \\ &= s_1s_1^* \otimes \mathbf{u} + s_2s_2^* \otimes 1 \\ &= s_1s_1^* \otimes \mathbf{u} + s_2s_1s_1^*s_2^* \otimes 1 + s_2s_2s_2^*s_2^* \otimes 1 && \text{because } 1_{\mathcal{O}_2} = s_1s_1^* + s_2s_2^* \\ &= V_1\sigma(\mathring{u})V_1^* + V_2V_2^* \\ &\sim_h V_1\sigma\pi(\tilde{u})V_1^* + V_2\sigma\pi(1_{\mathcal{T}_2 *_C \mathcal{T}_2})V_2^* && \text{by assertion 3)} \\ &\sim_h \sigma\pi(1_{\mathcal{T}_2 *_C \mathcal{T}_2}) && \text{by assertion 2)} \\ &= 1_{\mathcal{O}_2 \otimes C^*(\mathbb{Z})} = \sigma(1_{\mathcal{O}_2 *_C C^*(\mathbb{Z})}) \end{aligned}$$

As a consequence, $\sigma(\mathbf{u}) \sim_h \sigma(s_1s_1^* + s_2\mathbf{u}^*s_2^*)$ in $\mathcal{U}(\mathcal{O}_2 \otimes C^*(\mathbb{Z}))$.

5) The isomorphism $\mathcal{O}_2 *_C C^*(\mathbb{Z}) \cong \mathcal{O}_2 *_C \mathcal{O}_2$ induces two embeddings σ_1, σ_2 of the C^* -algebra $\mathcal{O}_2 *_C C^*(\mathbb{Z})$ into the free product $\mathcal{O}_2 *_C C^*(\mathbb{Z}) *_C C^*(\mathbb{Z}) = \mathcal{O}_2 *_C C^*(\mathbb{F}_2)$, where \mathbb{F}_2 is the free group with 2 generators. If the unitary \mathring{u} is in $\mathcal{U}^0(\mathcal{O}_2 *_C C^*(\mathbb{Z}))$, then the product $w := \sigma_1(\mathring{u})\sigma_2(\mathring{u})\sigma_1(\mathring{u})^{-1}\sigma_2(\mathring{u})^{-1}$ belongs to $\mathcal{U}^0(\mathcal{O}_2 *_C C^*(\mathbb{F}_2))$. We shall show in three steps that $w \notin \mathcal{U}^0(\mathcal{O}_2 *_C C^*(\mathbb{F}_2))$, and this will imply that the two unitaries \mathbf{u} and $s_1s_1^* + s_2\mathbf{u}^*s_2^*$ are not homotopic in the compact group $\mathcal{U}(\mathcal{O}_2 *_C C^*(\mathbb{Z}))$.

Let β be the action of the group $S^1 = \mathbb{R}/\mathbb{Z}$ on the C^* -algebra \mathcal{O}_2 given by $\beta_t(s_k) = e^{2i\pi t}s_k$ ($k = 1, 2$). The subalgebra $A \subset \mathcal{O}_2$ of β -invariant elements is the closure of the growing sequence of matrix C^* -algebras A_n linearly generated by the elements $s_{i_1} \dots s_{i_n} s_{i_{n+1}}^* \dots s_{i_{2n}}^*$ and where each A_n embeds in A_{n+1} by $a \mapsto s_1 a s_1^*$. This sequence of monomorphisms extends to an endomorphism α on A . Call \check{A} the inductive limit of the system $A \xrightarrow{\alpha} A \xrightarrow{\alpha} \dots$ with corresponding embeddings $\mu_n : A \rightarrow \check{A}$ ($n \in \mathbb{N}$) and let $\check{\alpha} : \check{A} \rightarrow \check{A}$ be the unique automorphism satisfying $\check{\alpha}(\mu_n(a)) = \mu_n(\alpha(a))$.

*Step 1. The unitary w does not belong to the connected component $\mathcal{U}^0(A *_\mathbb{C} C^*(\mathbb{Z}))$.*

Proof: Define for all integer $n \geq 2$ the compact space $U(n)$ of unitaries in $M_n(\mathbb{C})$ and let D_n be the C^* -algebra $D_n = C(U(n) \times U(n), M_n(\mathbb{C}))$. As noticed by B. Blackadar in Barcelona ([Blac07]), a theorem by S. Araki, M. James and E. Thomas ([AJT60]) implies that $w \notin \mathcal{U}^0(A_n *_\mathbb{C} C^*(\mathbb{Z}))$. Hence, $w \notin \mathcal{U}^0(A *_\mathbb{C} C^*(\mathbb{Z}))$ by passage to the limit since all the connecting maps $A_n \rightarrow A_{n+1}$ are injective.

*Step 2. The unitary w is not in $\mathcal{U}^0(\ddot{A} \rtimes \mathbb{Z} *_\mathbb{C} C^*(\mathbb{Z}))$.*

Proof: The density of the subset $\cup_n \mu_n(A)$ in \ddot{A} implies that $w \notin \mathcal{U}^0(\ddot{A} *_\mathbb{C} C^*(\mathbb{Z}))$. The epimorphism $f \mapsto f(1)$ from $C(S^1) = C^*(\mathbb{Z})$ to \mathbb{C} induces a surjection from $\ddot{A} \rtimes_{\ddot{\alpha}} \mathbb{Z} *_\mathbb{C} C^*(\mathbb{Z})$ to the subalgebra $\ddot{A} *_\mathbb{C} C^*(\mathbb{Z})$ and so w could be in $\mathcal{U}^0(\ddot{A} \rtimes \mathbb{Z} *_\mathbb{C} C^*(\mathbb{Z}))$ only if w was already in $\mathcal{U}^0(A *_\mathbb{C} C^*(\mathbb{Z}))$.

*Step 3. The unitary w does not belong to the connected component $\mathcal{U}^0(\mathcal{O}_2 *_\mathbb{C} C^*(\mathbb{Z}))$.*

Proof: If $p \in \ddot{A}$ is the projection $p = \mu_0(1_A)$, then $p(\ddot{A} \rtimes_{\ddot{\alpha}} \mathbb{Z})p \cong \mathcal{O}_2$ ([Cun77, Subsection 2.1], [DyShli01, Claim 3.4]). Hence the relation $w \notin \mathcal{U}^0(\ddot{A} \rtimes \mathbb{Z} *_\mathbb{C} C^*(\mathbb{Z}))$ implies that $w \notin \mathcal{U}^0(\mathcal{O}_2 *_\mathbb{C} C^*(\mathbb{Z}))$.

6) Assertions 3) and 5) of the present Proposition imply that the K_1 -trivial unitary \tilde{u} is not in $\mathcal{U}^0(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2)$. Proposition 3.3 from [Blan16] implies that the unital free product $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$ is not K_1 -injective. \square

Remarks 3. 1) The unital free product $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$ does not have real rank 0. Indeed, this would imply by Corollary 4.2.10 of [Lin01] that the C^* -algebra $\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2$ is K_1 -injective.
2) The C^* -algebra \mathcal{T}_2 is K_1 -injective and so the amalgamated free product $\mathcal{U}(\mathcal{T}_2) \underset{S^1}{*} \mathcal{U}(\mathcal{T}_2)$ embeds in the connected component $\mathcal{U}^0(\mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2)$.

Corollary 4. There exists a unital continuous $C([0, 1])$ -algebra with properly infinite fibres which is not a properly infinite C^* -algebra.

Proof. The C^* -algebra $\mathcal{D} := \{f \in C([0, 1], \mathcal{T}_2 *_\mathbb{C} \mathcal{T}_2); f(0) \in \mathcal{J}_0(\mathcal{T}_2) \text{ and } f(1) \in \mathcal{J}_1(\mathcal{T}_2)\}$ is a unital continuous $C([0, 1])$ -algebra with properly infinite fibres. Proposition 3.3 of [Blan16] and the above Proposition 2 imply that this C^* -algebra \mathcal{D} is not properly infinite.

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